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SCHOOL: SCIENCE AND TECHNLOGY

MODULE: MATHEMATICAL LOGIC

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Notes on mathematical logic

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Mathematical Logic

- **Propositional Logic** (Today)
- Basic logical connectives.
- Truth tables.
- Logical equivalences.
- **First-Order Logic**
- Reasoning about properties of multiple objects.

Propositional Logic

A **proposition** is a statement that is, by itself, either true or false.

Propositional Logic

- **Propositional logic** is a mathematical system for reasoning about propositions and how they relate to one another.
- Propositional logic enables us to
- Formally encode how the truth of various propositions influences the truth of other propositions.
- Determine if certain combinations of propositions are always, sometimes, or never true.
- Determine whether certain combinations of propositions logically entail other combinations.

Variables and Connectives

- Propositional logic is a formal mathematical system whose syntax is rigidly specified.
- Every statement in propositional logic consists of **propositional variables** combined via **logical connectives**.
- Each variable represents some proposition, such as
  "You wanted it" or "You should have put a ring on it."
- Connectives encode how propositions are related, such as "If you wanted it, you should have put a ring on it."

Propositional Variables

- Each proposition will be represented by a **propositional variable**.
- Propositional variables are usually represented as lower-case letters, such as *p*, *q*, *r*, *s*, etc.
- If we need more, we can use subscripts: p_1, p_2, etc.
- Each variable can take one one of two values: true or false.

Logical Connectives

- **Logical NOT: ¬*p***
  - Read "**not** *p*"
  - ¬*p* is true if and only if p is false.
  - Also called **logical negation**.
- **Logical AND: *p* ∧ *q***
  - Read "*p* **and** *q*."
  - *p* ∧ *q* is true if both p and q are true.
  - Also called **logical conjunction**.

- **Logical OR: $p \lor q$**
  - Read "*p* **or** *q*."
  - $p \lor q$ is true if at least one of *p* or *q* are true (inclusive OR)
  - Also called **logical disjunction**.

Truth Tables

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	q	$p \vee q$

Truth Tables

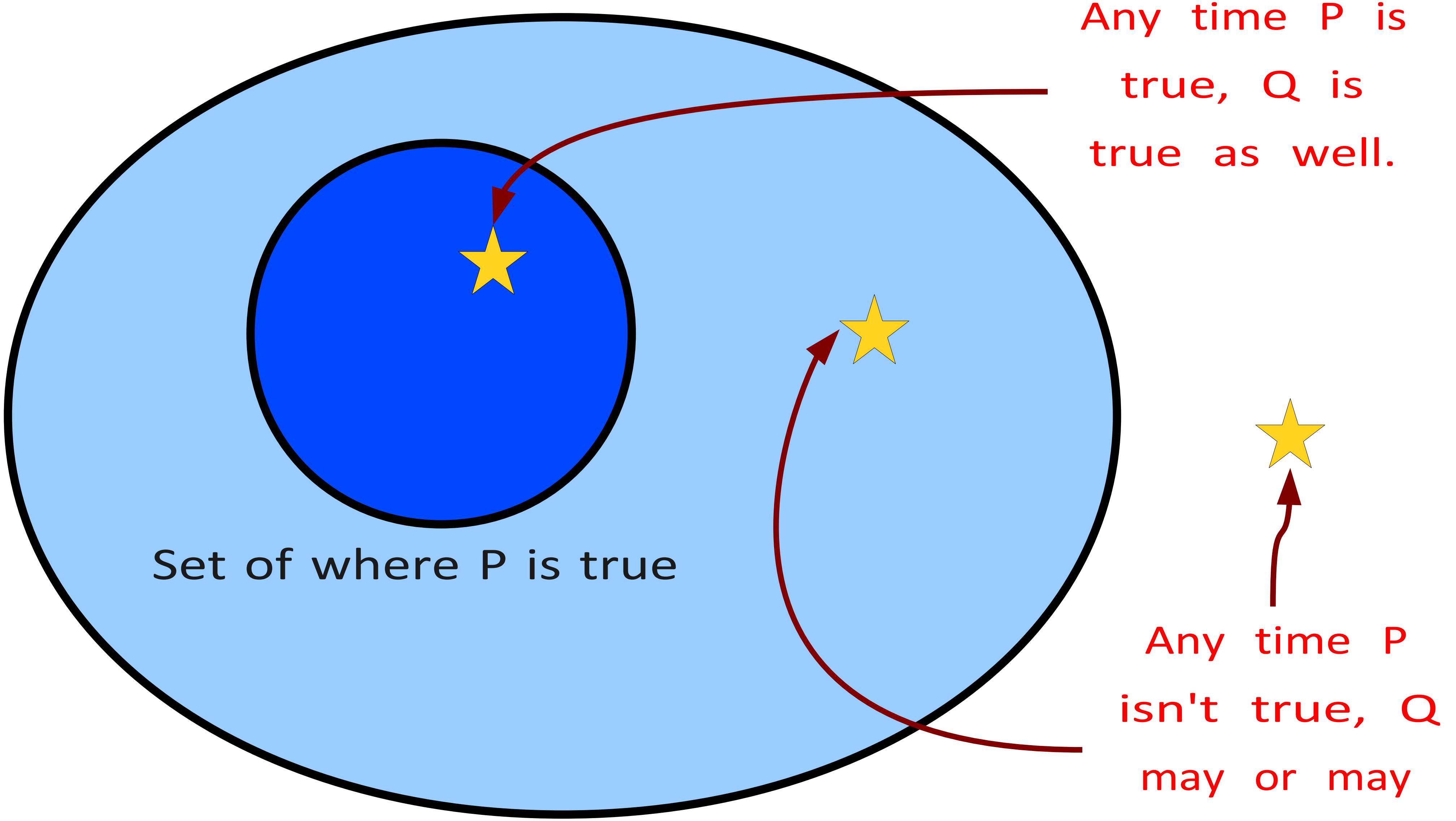
F	F	F
F	T	T
T	F	T
T	T	T

p	$\neg p$ F
T	T F

Implication

- An important connective is logical implication: $p \rightarrow q$.
- Recall: $p \rightarrow q$ means “if p is true, q is true as well.”
- Recall: $p \rightarrow q$ says **nothing** about what happens if p is false.
- Recall: $p \rightarrow q$ says **nothing** about causality; it just says that if p is true, q will be true as well.

Implication, Diagrammatically

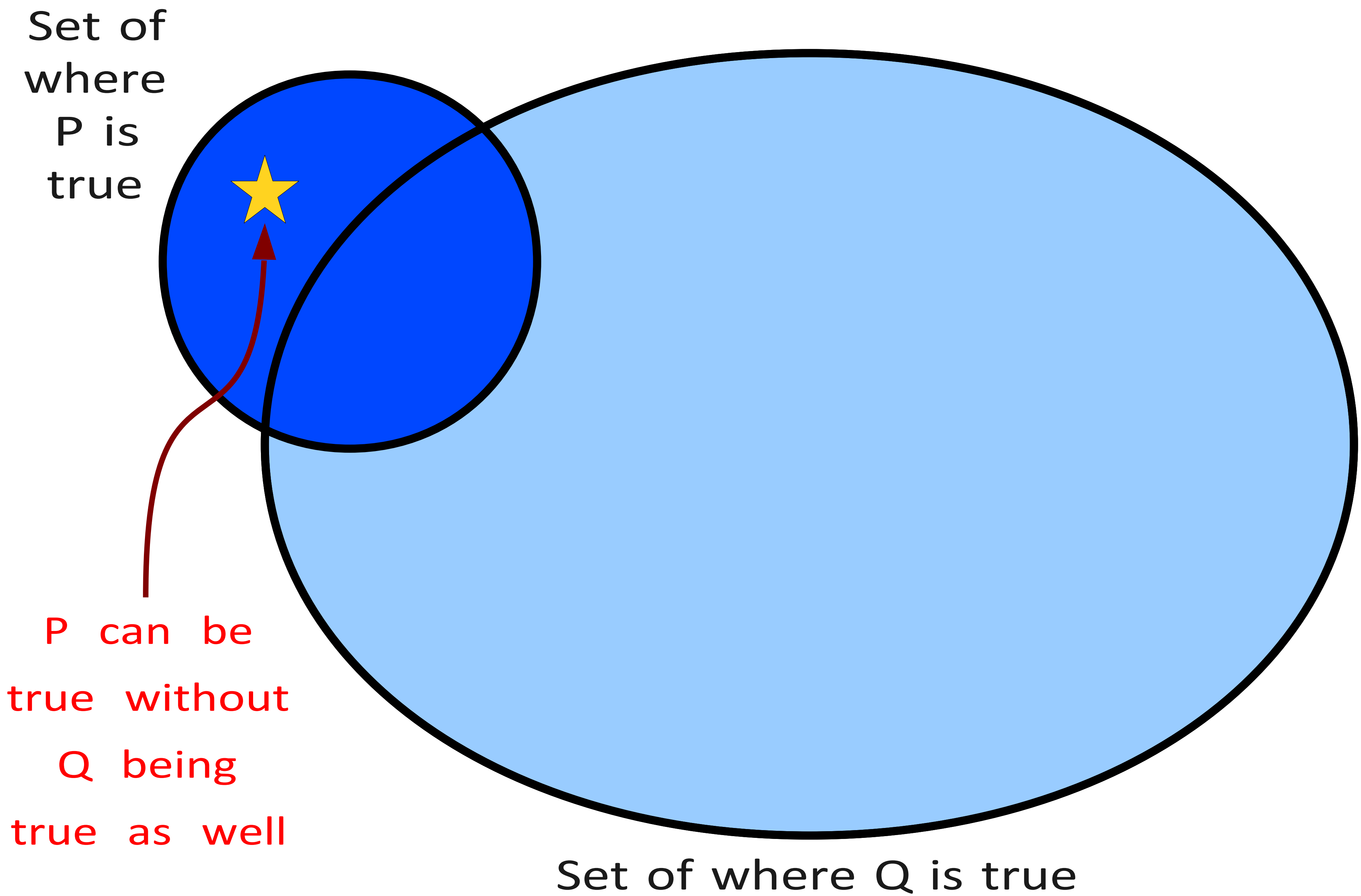


Set of where Q is true

not be true.

When *p* Does Not Imply *q*

- *p* → *q* means “if p is true, q is true as well.”
- Recall: The **only way** for *p* → *q* to be false is if we know that *p* is true but *q* is false.
- Rationale:
- If *p* is false, *p* → *q* doesn't guarantee anything. It's true, but it's not **meaningful**.
- If *p* is true and *q* is true, then the statement “if *p* is true, then *q* is also true” is itself true.
- If *p* is true and *q* is false, then the statement “if *p* is true, *q* is also true” is false.



Set of where P is true
P can be true without Q being true as well
Set of where Q is true

P

→ Q is false

Truth Table for Implication

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

The Biconditional

- The **biconditional** connective $p \leftrightarrow q$ is read "*p* if and only if *q*."
- Intuitively, either both *p* and *q* are true, or neither of them are.

p	q	$p \leftrightarrow q$
F	F	T
F	T	
T	F	F
T	T	T

One interpretation of is ↔

to think of it as equality:

F the two propositions must have equal truth values.

True and False

- There are two more “connectives” to speak of: true and false.
- The symbol $\top$ is a value that is always true.
- The symbol $\bot$ is value that is always false.
- These are often called connectives, though they don't connect anything.
- (Or rather, they connect zero things.)
-

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow ((y \lor z) \rightarrow (x \lor (y \land z)))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Recap So Far

- A **propositional variable** is a variable that is either true or false.
- The **logical connectives** are
- Negation: $\neg p$
- Conjunction: $p \land q$
- Disjunction: $p \lor q$
- Implication: $p \to q$
- Biconditional: $p \leftrightarrow q$
- True: $\top$
- False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: There is a velociraptor outside my apartment. *b*: Velociraptors can open windows. *c*: I am in my apartment right now. *d*: My apartment has windows.

e: I am going to be eaten by a velociraptor

I won't be eaten by a velociraptor if there isn't a velociraptor outside my apartment.

$$\neg a \rightarrow \neg e$$

“p if q”

translates to $q \rightarrow p$

It does **not** translate to $p \rightarrow q$

a: There is a velociraptor outside my apartment. *b*: Velociraptors can open windows. *c*: I am in my apartment right now. *d*: My apartment has windows.

Some Sample Propositions

e: I am going to be eaten by a velociraptor If there is a velociraptor outside my apartment, but it can't open windows, I am not going to be eaten by a velociraptor. $a \wedge \neg b \rightarrow \neg e$

"p, but q"

translates to $p \wedge q$

a: There is a velociraptor outside my apartment. *b*: Velociraptors can open windows. *c*: I am in my apartment right now. *d*: My apartment has windows. *e*: I am going to be eaten by a velociraptor

I am only in my apartment when there are no velociraptors outside.

Some Sample Propositions

$$c \rightarrow \neg a$$

"*p* only when *q*"

translates to $p \rightarrow q$

The Takeaway Point

- When translating into or out of propositional logic, be very careful not to get tripped up by nuances of the English language.
- In fact, this is one of the reasons we have a symbolic notation in the first place!
- Many prepositions lead to counterintuitive translations; make sure to double-check yourself!

Logical Equivalence

More Elaborate Truth Tables

This gives the final truth value for the

expression.

p	q		$p \wedge (p \to q)$
F	F		F T
F	T	F T	T F F F T T T T

Negations

- $p \land q$ is false if and only if $\neg(p \land q)$ is true.
- Intuitively, this is only possible if either p is false or q is false (or both!)
- In propositional logic, we can write this as $\neg p \lor \neg q$.
- How would we prove that $\neg(p \land q)$ and $\neg p \lor \neg q$ are equivalent?
- **Idea**: Build truth tables for both expressions and confirm that they always agree.

Negating AND

p	q	$\neg$	$(p \wedge q)$
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

p	q	$\neg p$	$\vee$	$\neg q$
F	F	T	T	T
F	T	T	T	F
T	F	F	T	T
T	T	F	F	F

These two statements are always the same!

Logical Equivalence

- If two propositional logic statements ϕ and ψ always have the same truth values as one another, they are called **logically equivalent**.
- We denote this by **$\phi \equiv \psi$**.
- ≡ is **not** a connective. Connectives are a part of logic statements; ≡ is something used to describe logic statements.
- It is part of the **metalanguage** rather than the **language**.
- If $\phi \equiv \psi$, we can modify any propositional logic formula containing ϕ by replacing it with ψ.
- This is not true when we talk about first-order logic; we'll see why later.

De Morgan's Laws

- Using truth tables, we concluded that

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

- We can also use truth tables to show that

$$\neg(p \lor q) \equiv \neg p \land \neg q$$

- These two equivalences are called **De Morgan's Laws**.

More Negations

- When is $p \rightarrow q$ false?
- **Answer**: p must be true and q must be false.
- In propositional logic: $p \land \neg q$
- Is the following true?

$$\neg(p \rightarrow q) \equiv p \land \neg q$$

Negating Implications

p q $\neg(p \to q)$

FT	F	F	T
TF	T	F	T
	F	T	F
	T	F	T

p q $p \land \neg q$

F	F	F	F	FF TT
F	T	F	F	
T	F	T	T	
T	T	T	F	

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

An Important Observation

- We have just proven that

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

- If we negate both sides, we get that $p \rightarrow q \equiv \neg(p \wedge \neg q)$.
- By De Morgan's laws: $p \rightarrow q \equiv \neg(p \wedge \neg q)$ $p \rightarrow q \equiv \neg p \vee \neg\neg q$ $p \rightarrow q \equiv \neg p \vee q$
- Thus $p \rightarrow q \equiv \neg p \vee q$

Another Idea

- We've just shown that $\neg(p \to q) \equiv p \land \neg q$.
- Is it also true that $\neg(p \to q) \equiv p \to \neg q$?
- Let's go check!

$\neg(p \rightarrow q)$ and $p \rightarrow \neg q$

p q $\neg(p \rightarrow q)$

F F F T

F T F T F T F T F T F T F T F T T T

T T F T

p q $p \rightarrow \neg q$

F F F T T

T T T F F

These are ***not*** the same thing!

To prove that $p \rightarrow q$ is false, do **not** prove $p \rightarrow \neg q$.

Instead, prove that $p \wedge \neg q$ is true.

Analyzing Proof Techniques

Proof by Contrapositive

- Recall that to prove that $p \rightarrow q$, we can also show that $\neg q \rightarrow \neg p$.
- Let's verify that $p \rightarrow q \equiv \neg q \rightarrow \neg p$.

The Contrapositive

$p \quad q \quad p \rightarrow q$

FT	F		T
FT	T		T
	F		F
	T		T

$p \quad q \quad \neg q \rightarrow \neg p$

F	F	T	T	
F	T	F	T	TF
T	F	T	F	
T	T	F	T	

TF $p \rightarrow q \equiv \neg q \rightarrow \neg p$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$" $x < 8 \land y < 8 \rightarrow$

$x + y \neq 16$

“If x < 8 and y < 8, then x + y ≠ 16” *Theorem:* If $x + y = 16$, then either $x \geq 8$ or $y \geq 8$.

Proof: By contrapositive. We prove that if $x < 8$ and $y < 8$, then $x + y \neq 16$. To see this, note that

$$\begin{aligned} x + y &< 8 + y \\ &< 8 + 8 \\ &= 16 \end{aligned}$$

So $x + y < 16$, so $x + y \neq 16$. ■

Why This Matters

- Propositional logic is a tool for reasoning about how various statements affect one another.
- To better understand how to prove a result, it often helps to translate what you're trying to prove into propositional logic first.
- **Note**: To truly reason about proofs, we need the more expressive power of **first-order logic**, which we'll talk about next time.

Proof by Contradiction

- The general structure of a proof by contradiction is
- To show *p*, assume *p* is false.
- Show that *p* being false implies something that cannot be true.
- Conclude, therefore, that p is true.
- What does this look like in propositional logic?

$$(\neg p \rightarrow \bot) \rightarrow p$$

Proof by Contradiction

p	$(\neg p$	$\rightarrow$	$\bot)$	$\rightarrow$	p
F	T	F	F	T	F
T	F	T	F	T	T

This statement is <u>always</u> true!

Tautologies

- A **tautology** is a statement that is always true.
- Examples:
-
- $p \lor \neg p$ (the **Law of the Excluded Middle**)
- $\bot \to p$ (**vacuous truth**)
- Once a tautology has been proven, we can use that tautology anywhere.
  -
  -
  -
  - **First-Order Logic**

- How do we reason about multiple objects and their properties?